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T–DUALITY AND WORLD–SHEET SUPERSYMMETRY

Ioannis Bakas * †

Theory Division, CERN
CH-1211 Geneva 23, Switzerland

and

Konstadinos Sfetsos ‡

Institute for Theoretical Physics, Utrecht University
Princetonplein 5, TA 3508, The Netherlands

ABSTRACT

Four–dimensional string backgrounds with local realizations of $N = 4$ world–sheet supersymmetry have, in the presence of a rotational Killing symmetry, only one complex structure which is an $SO(2)$ singlet, while the other two form an $SO(2)$ doublet. Although $N = 2$ world–sheet supersymmetry is always preserved under Abelian T–duality transformations, $N = 4$ breaks down to $N = 2$ in the rotational case. A non–local realization of $N = 4$ supersymmetry emerges, instead, with world–sheet parafermions. For $SO(3)$–invariant metrics of purely rotational type, like the Taub–NUT and the Atiyah–Hitchin metrics, none of the locally realized extended world–sheet supersymmetries can be preserved under non–Abelian duality.

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*Permanent address: Department of Physics, University of Patras, GR–26110 Patras, Greece
†e–mail address: BAKAS@SURYA11.CERN.CH
‡e–mail address: SFETSOS@FYS.RUU.NL

The duality symmetries of string theory describe a quantum equivalence between the underlying conformal field theories of various backgrounds with different geometrical (or even topological) properties. The target space duality between string backgrounds with Abelian isometries is the most notable example, known as T–duality (see, for instance, [1] and references therein). The toroidal compactification of the heterotic string theory seems to exhibit another remarkable symmetry under an $SL(2, Z)$ group of transformations that act on the coupling constant of the theory. This is the S–duality and it is more conveniently described as a symmetry of the axion–dilaton system in the Einstein frame of string theory [2, 3]. It is natural, once both types of duality symmetries exist in a string theory, to intertwine them in order to construct new discrete symmetries [4, 5, 6]. It has recently been recognized, however, that such a procedure is not always compatible with the $N = 4$ world–sheet supersymmetry of superconformal string vacua [6]. The condition that characterizes the occurrence of such an obstruction was linked to the nature of the Abelian isometries of the corresponding string vacua.

The purpose of the present work is to provide a natural explanation of this obstruction by considering the explicit behaviour of the locally realized extended world–sheet supersymmetries under generic T–duality transformations. As we will see later, the local realizations of $N = 4$ world–sheet supersymmetry cannot be preserved by performing T–duality transformations with respect to rotational Killing vector fields, due to the peculiar behaviour of the three underlying independent complex structures. This problem does not arise for translational Killing symmetries. Hence, the standard arguments that relate the $N = 4$ world–sheet supersymmetry to the geometrical properties of the target space manifold will not be applicable after dualizing with respect to rotational Killing symmetries. The rotational Killing symmetries also spoil the standard supersymmetry transformations in the target space after T–duality [6, 7], but we will not elaborate on this issue. We will only remark at the end that the space–time supersymmetry may be realized differently in terms of the new target space fields after a rotational T–duality transformation. The question we are addressing here is also interesting for exploring the supersymmetric properties of the $O(d, d)$ deformations of a given conformal field theory background, in general, and in particular along the $J\bar{J}$–line of marginal deformations [8] (see also [9]).

The behaviour of extended world–sheet supersymmetry under T–duality transformations is important for understanding the sense in which target space dualities are string symmetries. It is true that the $N = 2$ world–sheet supersymmetry remains local under generic Abelian T–duality transformations. This problem was investigated in the literature before by constructing explicitly the relevant complex structure in the dual formulation (see [10] and references therein). On target spaces with torsion, there are two complex structures, a left and a right one, which are relevant for $N = 2$ supersymmetry. When the torsion is zero the left and the right complex structures are identical, but they transform differently under Abelian T–duality transformations. This issue and the properties of their commutator have already been considered in the most general case and we will not deal with them further. Our question is more elementary and refers to

the local properties of the three independent complex structures (either the left or the right) in superconformal theories with $N = 4$ supersymmetry. Such complications are certainly not present in theories with only $N = 2$ supersymmetry.

The target space metric in $N = 4$ superconformal theories with torsion is constrained to satisfy the conditions

$$g = \Omega g' \ , \qquad \Box' \Omega = 0 \ , \tag{1}$$

where g' is a hyper–Kahler metric [11] (see also [12]). This theorem is true provided that the three underlying complex structures are locally realized, in which case they are covariantly constant (including torsion). Recall that there are examples of $N = 4$, $\hat{c} = 4$ models where the above geometric conditions are not satisfied; in all such models, the $N = 4$ supersymmetry is non–locally realized on the world–sheet [13]. We will demonstrate that T–duality with respect to rotational Killing symmetries always leads to such a peculiar world–sheet behaviour. For example, the string background which is dual to flat space with respect to any one of its rotational Killing symmetries can be easily shown to contradict the conditions (1). It is also conceivable that all models with non–local realizations of $N = 4$ supersymmetry may admit a local description, in an appropriately chosen T–dual formulation, with respect to some rotational Killing symmetry.

In the following, we simplify our presentation by considering first the supersymmetric behaviour of pure gravitational string backgrounds under generic T–duality transformations and review some of the basic concepts. The geometric description of the Abelian duality in terms of canonical transformations in the target space [14, 15] will be particularly useful in finding the explicit form of the dual complex structures. We will also discuss these issues for some more general string backgrounds, including the worm–hole solution [12, 16]. Finally we will conclude with some generic features of non–Abelian duality and supersymmetry.

Four–dimensional pure gravitational backgrounds with $N = 4$ world–sheet supersymmetry are known to be hyper–Kahler manifolds (i.e. $\Omega = 1$ in eq. (1)). Their Riemann tensor satisfies the self–duality conditions

$$R_{\mu\nu\rho\sigma} = \pm \frac{1}{2} \sqrt{\det G} \ \epsilon_{\rho\sigma}{}^{\kappa\lambda} R_{\mu\nu\kappa\lambda} \tag{2}$$

with the $\pm$ sign corresponding to self–dual or anti–self–dual metrics respectively, depending on the conventions. If these backgrounds also admit (at least) one Killing symmetry, then, in the special coordinate system where the Killing symmetry becomes manifest, their metric will take a particularly simple form. There are two kinds of Killing symmetries that are in fact distinct from each other. The first kind, which is usually called translational, corresponds to Killing vector fields K_ν with self–dual covariant derivatives,

$$\nabla_\mu K_\nu = \pm \frac{1}{2} \sqrt{\det G} \ \epsilon_{\mu\nu}{}^{\rho\sigma} \nabla_\rho K_\sigma \ , \tag{3}$$

according to the two cases (2) respectively [17]. The translational symmetries are also known as triholomorphic, because of their special character in Kahlerian coordinates. The

second kind, which is usually called rotational, encompasses all other Killing vector fields. It is true that rotational symmetries are more rare than translational ones, although simple examples of self–dual metrics admit both; these include the flat space, the Taub–NUT and the Eguchi–Hanson gravitational instanton.

In the case of 4–dim hyper–Kahler manifolds with a translational Killing symmetry, the metric assumes the form

$$ds^2 = V(dT + \Omega_1 dX + \Omega_2 dY + \Omega_3 dZ)^2 + V^{-1}(dX^2 + dY^2 + dZ^2) \ , \tag{4}$$

where T is a coordinate adapted for the translational Killing vector field $\partial/\partial T$. Moreover, Ω_i are constrained to satisfy the special conditions

$$\partial_i V^{-1} = \pm \frac{1}{2} \epsilon_{ijk} (\partial_j \Omega_k - \partial_k \Omega_j) \ , \tag{5}$$

depending on the self–dual or the anti–self–dual character of the metric respectively [18]. It also follows that V^{-1} satisfies the 3–dim flat space Laplace equation. Localized solutions of this equation correspond to the familiar series of multi–centre Eguchi–Hanson gravitational instantons or to the multi–Taub–NUT family, depending on the asymptotic conditions on V^{-1} (see, for instance, [19] and references therein).

The three independent complex structures associated to self–dual metrics with translational symmetry have been explicitly constructed in the literature [20]. In the special coordinate system (4) they assume the particularly simple form

$$F_1 = (dT + \Omega_2 dY) \wedge dX - V^{-1} dY \wedge dZ \ , \tag{6}$$

$$F_2 = (dT + \Omega_1 dX) \wedge dY + V^{-1} dX \wedge dZ \ , \tag{7}$$

$$F_3 = (dT + \Omega_1 dX + \Omega_2 dY) \wedge dZ - V^{-1} dX \wedge dY \ , \tag{8}$$

where Ω_3 has been set equal to zero by a gauge transformation. It is straightforward to verify that they are covariantly constant on–shell and satisfy the $SU(2)$ Clifford algebra. The property that is important for our purposes is that all three complex structures remain invariant under T–shifts. This will be crucial for understanding the way the local realizations of $N = 4$ world–sheet supersymmetry behave, in a string setting, under T–duality transformations. It is also useful to note at this point that $S_{\pm} = b \pm V$, where b is the nut potential of the metric (4), is constant for self–dual or anti–self–dual metrics respectively [6].

In the case of 4–dim hyper–Kahler manifolds with a rotational Killing symmetry, there exists a coordinate system (τ, x, y, z) in which the corresponding line element takes the form

$$ds^2 = v(d\tau + \omega_1 dx + \omega_2 dy)^2 + v^{-1} \left(e^{\Psi} dx^2 + e^{\Psi} dy^2 + dz^2 \right) . \tag{9}$$

In these adapted coordinates the rotational Killing vector field is $\partial/\partial\tau$ and all the components of the metric are expressed in terms of a single scalar field $\Psi(x, y, z)$ [17], so that

$$v^{-1} = \partial_z \Psi \ , \quad \omega_1 = \mp \partial_y \Psi \ , \quad \omega_2 = \pm \partial_x \Psi \ , \tag{10}$$

where $\Psi(x,y,z)$ satisfies the continual Toda equation:

$$(\partial_x{}^2+\partial_y{}^2)\Psi+\partial_z{}^2e^{\Psi}=0\ . \tag{11}$$

These coordinates actually provide the geodesic form of the corresponding 3–dim line element for which $S_\pm=b\pm v=-z$, and hence is not constant [6]. We use capital (small) letters to distinguish the special coordinates in the presence of translational (rotational) Killing symmetries.

Metrics with rotational Killing symmetry differ from those with translational symmetry in that not all three independent complex structures can be chosen to be τ–shift invariant. In fact, only one complex structure can be chosen to be an $SO(2)$ singlet, while the other two necessarily form an $SO(2)$ doublet. We have explicitly

$$F_3=(d\tau+\omega_1dx+\omega_2dy)\wedge dz+v^{-1}e^{\Psi}dx\wedge dy \tag{12}$$

for the singlet and

$$\begin{pmatrix} F_1 \\ \\ F_2\end{pmatrix}=e^{\frac{1}{2}\Psi}\begin{pmatrix}\cos\frac{\tau}{2} & \sin\frac{\tau}{2} \\ \\ \sin\frac{\tau}{2} & -\cos\frac{\tau}{2}\end{pmatrix}\begin{pmatrix} f_1 \\ \\ f_2\end{pmatrix} \tag{13}$$

for the doublet, where

$$f_1=(d\tau+\omega_2dy)\wedge dx-v^{-1}dz\wedge dy\ , \tag{14}$$

$$f_2=(d\tau+\omega_1dx)\wedge dy+v^{-1}dz\wedge dx\ . \tag{15}$$

It can be easily verified, once the right guess has been made, that F_1, F_2 and F_3 are covariantly constant on–shell (11) and satisfy the $SU(2)$ Clifford algebra *. The general explicit construction of all three complex structures in the rotational frame (9) has not appeared in the literature before, to the best of our knowledge. The expressions (12)–(15) are strictly speaking correct only for self–dual metrics. Their anti–self–dual counterparts, together with the metric (9), can simply be obtained using the substitution $\tau\to-\tau$.

It is possible to guess the general form of the three complex structures in the rotational frame by considering the special example of the Eguchi–Hanson gravitational instanton. This background is $SO(3)$–symmetric, and with respect to one of its three translational Killing symmetries it can be written in the form (4), where

$$V^{-1}=\frac{1}{R_+}+\frac{1}{R_-}\ ,\quad \Omega_1=-\frac{Y}{X^2+Y^2}\left(\frac{Z_+}{R_+}+\frac{Z_-}{R_-}\right),\quad \Omega_2=\frac{X}{X^2+Y^2}\left(\frac{Z_+}{R_+}+\frac{Z_-}{R_-}\right), \tag{16}$$

with $\Omega_3=0$ and

$$Z_\pm=Z\pm a\ ,\qquad R_\pm^2=X^2+Y^2+Z_\pm^2\ . \tag{17}$$

*It is known that the continual Toda equation (11) exhibits a W_∞ symmetry on–shell [21]. This symmetry preserves the sphere of complex structures defined by F_1, F_2 and F_3.

The moduli parameter is a. The Eguchi–Hanson instanton also has an additional $U(1)$ Killing symmetry which is rotational. The coordinate transformation that makes the latter manifest is

$$\begin{pmatrix} X \\ \\ Y \end{pmatrix} = e^{\frac{1}{2}\Psi} \begin{pmatrix} \cos\frac{\tau}{2} & \sin\frac{\tau}{2} \\ & \\ \sin\frac{\tau}{2} & -\cos\frac{\tau}{2} \end{pmatrix} \begin{pmatrix} x \\ \\ y \end{pmatrix} , \quad Z = z \, \frac{1 - \frac{1}{8}(x^2+y^2)}{1+\frac{1}{8}(x^2+y^2)} , \quad T = 2 \, \tan^{-1}\frac{y}{x} , \tag{18}$$

where

$$\Psi(x,y,z) = \log \frac{z^2 - a^2}{2\left(1 + \frac{1}{8}(x^2+y^2)\right)^2} \tag{19}$$

is the corresponding Toda potential. It follows that the three independent complex structures (6)–(8) are mapped to (12)–(15) by the coordinate transformation described above. Similar remarks apply to the flat space metric, which has $\Psi(x,y,z) = \log z$ in the rotational frame (9).

It is useful to recall, before we proceed further, that the Taub–NUT and the Atiyah–Hitchin metric on the moduli space of the $SU(2)$ 2–monopole solutions (in the BPS limit) [22] provide another two non–trivial examples of 4–dim hyper–Kahler manifolds with rotational Killing symmetries. Both of them admit an $SO(3)$ isometry of purely rotational type, but the Taub–NUT metric also admits an additional $U(1)$ symmetry which is translational. (This should be compared with the opposite character of the Eguchi–Hanson Killing symmetries.) Moreover, this completes the classification of the complete $SO(3)$–invariant self–dual metrics [20]. The Atiyah–Hitchin metric can be regarded as the simplest example of hyper–Kahler manifolds with only rotational Killing symmetries. Since the existence of only two rotational symmetries (and no other of either type) is not allowed from general considerations [17], it follows that the next more complicated solutions of this kind (if they exist at all) will exhibit only one rotational Killing symmetry and no other symmetries of either type. Work in this direction is in progress, while determining the Toda potential that corresponds to the Atiyah–Hitchin metric, its possible generalization to a new series of rotational instantons, and its use as a supersymmetric gravitational string background [23].

We have already seen that, in metrics with (at least) one Killing symmetry, either all complex structures behave as singlets or one of them is a singlet and the remaining two form an $SO(2)$ doublet. This applies to the conformal class of 4–dim metrics (1) and we will next investigate the effect of the T–duality on them. We first recall the essential ingredients for describing the Abelian T–duality transformation as a canonical transformation in the target space [15] (see also [14] for some earlier ideas). The classical propagation of strings in a general target space with metric $G_{\mu\nu}(X)$ and antisymmetric tensor field $B_{\mu\nu}(X)$ is described by the 2–dim σ–model density $Q_{\mu\nu}\partial X^{\mu}\bar{\partial}X^{\nu}$, where

$$Q_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu} \ . \tag{20}$$

Consider backgrounds with a Killing symmetry generated by the vector field $\partial/\partial X^0$ (X^0 will be T or τ in the translational or in the rotational frame, respectively). Let P_0 be

the canonical momentum conjugate to the string variable X^0; we perform the canonical transformation $(X^0,\ P_0) \rightarrow (\tilde{X}^0,\ \tilde{P}_0)$, as is prescribed by the interchange $P_0 \leftrightarrow \partial_\sigma X^0$, where σ denotes the spatial coordinate on the string world–sheet. This is equivalent to the transformation

$$\partial X^0 \rightarrow \frac{1}{G_{00}}(\partial X^0 - Q_{i0}\partial X^i) \ , \qquad \bar{\partial} X^0 \rightarrow -\frac{1}{G_{00}}(\bar{\partial} X^0 + Q_{0i}\bar{\partial} X^i) \ , \tag{21}$$

using the expression for the canonical momentum P_0. It is then straightforward to read–off the form of the dual string background by substituting (21) into the Hamiltonian form of the 2–dim action. The result,

$$\tilde{G}_{00} = \frac{1}{G_{00}} \ , \quad \tilde{Q}_{0i} = \frac{Q_{0i}}{G_{00}} \ , \quad \tilde{Q}_{i0} = -\frac{Q_{i0}}{G_{00}} \ , \quad \tilde{Q}_{ij} = Q_{ij} - \frac{Q_{i0}Q_{0j}}{G_{00}} \ , \tag{22}$$

indeed describes the Abelian T–duality transformation in all generality. The conformal invariance also requires that the corresponding dilaton field 2Φ is shifted by $-\log G_{00}$ (see, for instance, [1]).

The transformation (21) amounts to a non–local redefinition of the target space variable associated with the Killing symmetry,

$$X^0 \rightarrow \int \frac{1}{G_{00}} \Big((\partial X^0 - Q_{i0}\partial X^i) dz - (\bar{\partial} X^0 + Q_{0i}\bar{\partial} X^i) d\bar{z} \Big) \ . \tag{23}$$

Despite the non–localities, the dual target space fields (22) are locally related to the original ones. However, other geometrical quantities in the target space, such as the Kahler 2–forms describing the complex structures, are not bound to be always local in the dual picture. This is precisely our concern for addressing the question of local versus non–local realizations of $N = 4$ world–sheet supersymmetry, in general, upon duality. We are considering the local behaviour of three independent almost complex structures, which are actually Hermitian, without worrying about their integrability, since the Nijenhuis conditions are not necessary for the existence of extended world–sheet supersymmetry [24].

The complex structures come in pairs in the presence of torsion and define the 2–forms

$$F_I^{\pm} = (F_I^{\pm})_{\mu\nu} dX^\mu \wedge dX^\nu = 2(F_I^{\pm})_{0i} dX^0 \wedge dX^i + (F_I^{\pm})_{ij} dX^i \wedge dX^j \tag{24}$$

that satisfy all the necessary conditions (they are covariantly constant, including the torsion, and each $+$ or $-$ set forms separately an $SU(2)$ Clifford algebra). The $F_I^{\pm}$ are associated to right or left–handed fermions and in order to find the correct transformation properties under T–duality, we simply have to use the replacement $dX^\mu \rightarrow \partial X^\mu$ for F_I^+ and $dX^\mu \rightarrow \bar{\partial} X^\mu$ for F_I^-. Of course, this is only meant to be a prescription for extracting the relevant part of the complex structures under the duality transformation (21). With this explanation in mind, we first consider the simplest case having all $(F_I^{\pm})_{\mu\nu}$ independent of X^0. The result we obtain for the dual complex structures in component form is

$$(\tilde{F}_I^+)_{0i} = \frac{1}{G_{00}}(F_I^+)_{0i} \ , \quad (\tilde{F}_I^+)_{ij} = (F_I^+)_{ij} + \frac{1}{G_{00}} \Big((F_I^+)_{0i} Q_{j0} - (F_I^+)_{0j} Q_{i0} \Big) \tag{25}$$

and

$$(\tilde{F}_I^-)_{0i} = -\frac{1}{G_{00}}(F_I^-)_{0i} \ , \quad (\tilde{F}_I^-)_{ij} = (F_I^-)_{ij} + \frac{1}{G_{00}}\Big((F_I^-)_{0i}Q_{0j} - (F_I^-)_{0j}Q_{0i}\Big) \ . \tag{26}$$

If we were computing $(\tilde{F}_I^{\pm})^{\mu}{}_{\nu} = \tilde{G}^{\mu\lambda}(\tilde{F}_I^{\pm})_{\lambda\nu}$ in this case, the result would coincide with the expressions for the dual complex stuctures derived before [10].

It follows from the previous analysis that all the components $(F_I^{\pm})_{\mu\nu}$ will be independent of X^0 if the Killing vector field $\partial/\partial X^0 \equiv \partial/\partial T$ is translational. We consider the effect of T–duality on the self–dual gravitational backgrounds (4), (5) as an application in this case. The dual background is conformally flat,

$$d\tilde{s}^2 = V^{-1}(dT^2 + dX^2 + dY^2 + dZ^2) \ , \tag{27}$$

with a non–trivial antisymmetric tensor field

$$\tilde{B} = 2 \ dT \wedge (\Omega_1 dX + \Omega_2 dY) \tag{28}$$

and dilaton field $2\tilde{\Phi} = \log V^{-1}$. The corresponding dual complex structures can be obtained from eqs. (6)–(8) and they assume the form

$$\tilde{F}_I^{\pm} = V^{-1}\left(\pm dT \wedge dX^I - \frac{1}{2}\epsilon_{IJK} dX^J \wedge dX^K\right) \ , \tag{29}$$

where $\{X^I\} = \{X, Y, Z\}$. The dual background is consistent with the condition (1), because the conformal factor $\Omega = V^{-1}$ satisfies the Laplace equation in flat space that was imposed by the self–duality of the original metric. The resulting backgrounds are the axionic instantons introduced in the toroidal compactification of the heterotic string theory [12, 25]. In this case, the solutions exhibit $N = 4$ world–sheet supersymmetry, which is locally realized and hence compatible with the geometric characterization (1) of the target space metric before and after duality.

The situation is radically different when the T–duality is performed with respect to a rotational Killing symmetry. A generic string background with locally realized $N = 4$ world–sheet supersymmetry has a hyper–Kahler metric g' associated with it, according to eq. (1). Using the rotational frame (9), in which $X^0 = \tau$, the T–duality transformation yields the background

$$d\tilde{s}^2 = v^{-1}(e^{\Psi}dx^2 + e^{\Psi}dy^2 + dz^2 + d\tau^2) \ , \tag{30}$$

with antisymmetric tensor field

$$\tilde{B} = 2 \ d\tau \wedge (\omega_1 dx + \omega_2 dy) \tag{31}$$

and dilaton field $2\tilde{\Phi} = \log v^{-1}$. The complex structure (12) will remain local in the dual picture, assuming the form

$$\tilde{F}_3^{\pm} = v^{-1}(\pm d\tau \wedge dz + e^{\Psi} dx \wedge dy) \ . \tag{32}$$

The 2–forms (14), (15) become similarly

$$\tilde{f}_1^{\pm} = v^{-1}(\pm d\tau \wedge dx - dz \wedge dy) \ , \quad \tilde{f}_2^{\pm} = v^{-1}(\pm d\tau \wedge dy + dz \wedge dx) \ . \tag{33}$$

The duality transformation amounts to the canonical transformation

$$\tau \rightarrow \int (v^{-1}\partial\tau - \omega_1 \partial x - \omega_2 \partial y) dz - (v^{-1}\bar{\partial}\tau + \omega_1 \bar{\partial} x + \omega_2 \bar{\partial} y) d\bar{z} \tag{34}$$

and so the components of the forms F_1 and F_2, which depend explicitly on τ via trigonometric functions, will become non–local after the rotational T–duality. Moreover, the resulting $\tilde{F}_1^{\pm}$ and $\tilde{F}_2^{\pm}$ are not covariantly constant on–shell, including the torsion coming from eq. (31).

We found that the local realization of $N = 4$ world–sheet supersymmetry breaks down to $N = 2$, with $\tilde{F}_3^{\pm}$ providing the relevant pair of complex structures in the presence of torsion. This is also consistent with the fact that the dual background (30) is not conformally equivalent to a hyper–Kahler metric, as would have been required otherwise by $N = 4$ world–sheet supersymmetry. It should be noted, though, that general arguments from superconformal field theory indicate that the $N = 4$ world–sheet supersymmetry will remain present, but part of it will become hidden into a non–local realization. We do not have an exact conformal field theory description of the string gravitational background (30), (31) in order to illustrate this point in all generality. For this reason we will examine the question in the special case of the 4–dim worm–hole solution and its rotational dual background, where an exact description is available in terms of the $SU(2)$ WZW model and its derivatives. We will see later that the parafermion currents of the $SU(2)/U(1)$ coset model describe the non–local structure of the dual 2–forms $\tilde{F}_1^{\pm}$, $\tilde{F}_2^{\pm}$, thus providing the explicit construction of a non–locally realized $N = 4$ superconformal algebra with $\hat{c} = 4$ [13].

An analogous situation arises when a string background exhibits a non–Abelian symmetry group. Various Killing symmetries that do not commute with a rotational isometry will become non–locally realized after performing the T–duality. They all remain symmetries of the dual model, but some of them are hidden in the non–local realization of the corresponding symmetry algebra. This issue is illustrated with a simple 2–dim example, which we give separately as an appendix.

The worm–hole solution of 4–dim string theory provides an exact conformal field theory background with $N = 4$ world–sheet supersymmetry [12, 13, 16]. The $N = 4$ superconformal algebra is locally realized in terms of four bosonic currents, three non–Abelian $SU(2)_k$ currents and one Abelian current with background charge $Q = \sqrt{2/(k+2)}$, so that the central charge is $\hat{c} = 4$. There are also four free–fermion superpartners and the solution is described by the $SU(2)_k \otimes U(1)_Q$ supersymmetric WZW model [†]. The background fields of this model are given in holomorphic target space coordinates

$$ds^2 \ = \ V^{-1}(du d\bar{u} + dw d\bar{w}) \ , \qquad V = u\bar{u} + w\bar{w} \ , \tag{35}$$

[†]The realization of the $N = 4$ superconformal algebra in terms of $SU(2)$ currents was first considered in ref. [26].

$$B = \frac{1}{2}V^{-1}(u\bar{u} - w\bar{w})\left(\frac{1}{u\bar{w}}du \wedge d\bar{w} + \frac{1}{\bar{u}w}d\bar{u} \wedge dw\right) , \tag{36}$$

with a non–trivial dilaton field $2\Phi = \log V^{-1}$. This background is conformally flat and it satisfies the condition (1) as required. We note for completeness that it is not T–dual to a pure gravitational self–dual background with respect to a translational Killing symmetry. We have also determined the three independent pairs of the underlying complex structures of the model,

$$F_1^+ = \frac{1}{2}V^{-1}(du \wedge dw + d\bar{u} \wedge d\bar{w}) , \quad F_1^- = \frac{i}{2}V^{-1}(du \wedge d\bar{w} - d\bar{u} \wedge dw) , \tag{37}$$

$$F_2^+ = \frac{i}{2}V^{-1}(du \wedge dw - d\bar{u} \wedge d\bar{w}) , \quad F_2^- = \frac{1}{2}V^{-1}(du \wedge d\bar{w} + d\bar{u} \wedge dw) , \tag{38}$$

$$F_3^+ = \frac{i}{2}V^{-1}(du \wedge d\bar{u} + dw \wedge d\bar{w}) , \quad F_3^- = \frac{i}{2}V^{-1}(-du \wedge d\bar{u} + dw \wedge d\bar{w}) , \tag{39}$$

which satisfy all the necessary conditions for having $N = 4$ supersymmetry.

It is known that the worm–hole solution is dual to the exact superconformal field theory based on the WZW model $SU(2)_k/U(1) \otimes U(1) \otimes U(1)_Q$ with $\hat{c} = 4$ [10, 13] (see also [27]). This can be demonstrated by introducing the polar coordinates ρ, τ, ψ and φ in the target space, so that

$$u = e^{\rho + i\tau} \cos\varphi , \qquad w = e^{\rho + i\psi} \sin\varphi , \tag{40}$$

and performing the T–duality transformation in the rotational τ direction. It is actually more convenient for calculational purposes to consider the target space variables

$$\alpha = \frac{1}{2}\psi - \tau , \qquad \beta = \frac{1}{2}\psi + \tau . \tag{41}$$

Then, the duality transformation reads

$$\tau \rightarrow \tilde{\tau} \equiv \int (\partial\beta - \tan^2\varphi \ \partial\alpha)dz - (\bar{\partial}\beta - \tan^2\varphi \ \bar{\partial}\alpha)d\bar{z} \tag{42}$$

and the resulting new background has no antisymmetric tensor field. The dual metric assumes the form

$$d\tilde{s}^2 = d\rho^2 + d\beta^2 + d\varphi^2 + \tan^2\varphi \ d\alpha^2 , \tag{43}$$

while the corresponding dilaton field is $-2\tilde{\Phi} = 2\rho + \log(\cos^2\varphi)$. This string background, which corresponds to the $SU(2)/U(1)_k \otimes U(1) \otimes U(1)_Q$ coset model, has the special feature that the metric (43) is not hyper–Kahler. Both conformal field theories admit an $N = 4$ superconformal symmetry, but it is not surprising that it is non–locally realized in the latter.

We will demonstrate that this non–local realization of the $N = 4$ world–sheet supersymmetry provides a simple example of our general framework. The complex structure, which is dual to (39), turns out to be local:

$$\tilde{F}_3 = d\rho \wedge d\beta + \tan\varphi \ d\varphi \wedge d\alpha , \tag{44}$$

and there is no distinction between the + and − components, since $\tilde{B} = 0$. The complex structures (37) and (38), on the other hand, become non–locally realized in the dual model. The resulting expressions, up to an irrelevant factor of 1/2, are

$$\begin{aligned} \tilde{F}_1^+ &= (d\rho + i\ d\beta) \wedge \Psi_+ + (d\rho - i\ d\beta) \wedge \Psi_- \ , \qquad (45) \\ \tilde{F}_1^- &= i(d\rho - i\ d\beta) \wedge \bar{\Psi}_+ - i(d\rho + i\ d\beta) \wedge \bar{\Psi}_- \qquad (46) \end{aligned}$$

and

$$\begin{aligned} \tilde{F}_2^+ &= i(d\rho + i\ d\beta) \wedge \Psi_+ - i(d\rho - i\ d\beta) \wedge \Psi_- \ , \qquad (47) \\ \tilde{F}_2^- &= (d\rho - i\ d\beta) \wedge \bar{\Psi}_+ + (d\rho + i\ d\beta) \wedge \bar{\Psi}_- \ , \qquad (48) \end{aligned}$$

where

$$\Psi_\pm = (d\varphi \pm i \tan\varphi\ d\alpha) e^{\pm i(\tilde{\tau}+\alpha+\beta)} \ , \quad \bar{\Psi}_\pm = (d\varphi \mp i \tan\varphi\ d\alpha) e^{\pm i(\tilde{\tau}-\alpha-\beta)} \qquad (49)$$

are non–local 1–forms with $\tilde{\tau}$ given by eq. (42). The above 2–forms are not covariantly constant on–shell and they are responsible for having a non–local realization of the $N = 4$ world–sheet supersymmetry in the dual to the worm–hole string model [13].

The non–local 1–forms (49) can be naturally decomposed into (1, 0) and (0, 1) forms on the string world–sheet,

$$\Psi_\pm = \Psi_\pm^{(1,0)} dz + \Psi_\pm^{(0,1)} d\bar{z} \ , \quad \bar{\Psi}_\pm = \bar{\Psi}_\pm^{(1,0)} dz + \bar{\Psi}_\pm^{(0,1)} d\bar{z} \ . \qquad (50)$$

It can be easily verified that

$$\bar{\partial} \Psi_\pm^{(1,0)} = 0 \ , \qquad \partial \bar{\Psi}_\pm^{(0,1)} = 0 \qquad (51)$$

are chirally conserved, using the classical equations of motion of the dual model, and in fact $\Psi_\pm^{(1,0)}$ and $\bar{\Psi}_\pm^{(0,1)}$ coincide with the classical parafermions that exist in this case; the field β actually provides the dressing of the $SU(2)/U(1)$ parafermions to the full 4–dim coset model. Hence, the usual $N = 4$ world–sheet supersymmetry of the 4–dim worm–hole background breaks down to $N = 2$ and a non–local realization of $N = 4$ emerges instead in the dual model with world–sheet parafermions. The T–duality transformation with respect to a rotational Killing symmetry is clearly the reason for this behaviour in string theory.

We conclude with a few remarks concerning the effect of the non–Abelian duality transformations on gravitational backgrounds with extended world–sheet supersymmetry. For $SO(3)$–invariant metrics, the complex structures either can be $SO(3)$ singlets, thus remaining invariant under the non–Abelian group action,

$$\pounds_J (F_I)_{\mu\nu} = 0 \ , \qquad (52)$$

or form an $SO(3)$ triplet when

$$\pounds_J (F_I)_{\mu\nu} = \epsilon_{JIK} (F_K)_{\mu\nu} . \qquad (53)$$

The Lie derivative is taken with respect to the $SO(3)$ generators of the isometry. The Eguchi–Hanson metric corresponds to the first case, while the Taub–NUT and the Atiyah–Hitchin metrics to the second [20]. Although there is no systematic formulation of the non–Abelian duality in terms of canonical transformations in the target space, we can easily find the dual complex structures for backgrounds where the original complex structures are $SO(3)$ singlets. In this case, the complex structures are written in terms of the left–invariant Maurer–Cartan forms of $SO(3)$. In the dual model they are obtained by replacing ordinary derivatives by covariant ones, using gauge fields, and then substituting the on–shell solution for the gauge fields and fixing a unitary gauge. Such an algorithm will certainly not be applicable if the complex structures form an $SO(3)$ triplet. According to this, the dual version of the Eguchi–Hanson instanton with respect to $SO(3)$ will have an $N = 4$ world–sheet supersymmetry locally realized. In fact, the non–Abelian duality has already been performed in this case, producing a conformally flat metric that satisfies the condition (1) [28]. On the other hand, performing the non–Abelian $SO(3)$–duality to the Taub–NUT and the Atiyah–Hitchin metrics will result in a total loss of all the locally realized extended world–sheet supersymmetries. We expect to have a non–local realization of supersymmetry in such cases with non–Abelian parafermions. Of course, the $N = 1$ supersymmetry will not be affected by any kind of T–duality transformations. Clearly, the relation of non–abelian duality with supersymmetry deserves a more thorough study.

In summary, we found that the T–duality transformations with respect to rotational Killing symmetries always break the local realizations of the $N = 4$ world–sheet supersymmetry to $N = 2$ (or even to $N = 1$ in the non–Abelian case). The effect of the intertwined T–S–T duality can be even more severe. For example, the T–S–T transformation of a purely gravitational background with $N = 4$ world–sheet supersymmetry will always yield a purely gravitational metric, which is Ricci–flat, but not Kahler, if T is rotational [6]. In this case, all the complex structures of the target spece manifold will be destroyed and therefore, no space–time supersymmetry can exist in the usual sense. These issues raise the question whether some new space–time supersymmetry generators can be defined by intertwining the standard ones with the rotational T–duality transformations,

$$\tilde{Q}_i = [T,\ Q_i]\ . \tag{54}$$

If this is possible, the $\tilde{Q}_i$ will have a very different realization in terms of the new background fields. The main example for this investigation is again provided by the coset model $SU(2)/U(1)_k \otimes U(1) \otimes U(1)_Q$. We hope to return to this problem elsewhere.

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APPENDIX

In this appendix we illustrate with a simple example the non–local realization of Killing symmetries after performing a rotational T–duality transformation. Similar ideas were presented by Kiritsis in the context of parafermionic symmetries [8] (see also [15]).

Consider the 2–dim action of two free bosons written in polar coordinates ρ, φ,

$$S = \int \partial\rho\bar{\partial}\rho + \rho^2 \partial\varphi\bar{\partial}\varphi \ . \tag{A.1}$$

This action exhibits isometries associated with the following variations,

$$\delta\varphi = \epsilon_0 + \epsilon_+ \frac{1}{\rho} e^{i\varphi} + \epsilon_- \frac{1}{\rho} e^{-i\varphi} \ , \qquad \delta\rho = -i\epsilon_+ e^{i\varphi} + i\epsilon_- e^{-i\varphi} \ , \tag{A.2}$$

where ϵ_0 and $\epsilon_\pm$ are constant. It can be verified that the corresponding vector fields

$$J_0 = i\partial_\varphi \ , \qquad J_\pm = e^{\mp i\varphi} \left(\frac{1}{\rho} \partial_\varphi \pm i\partial_\rho \right) \tag{A.3}$$

generate the Euclidean group in two dimensions, i.e. $[J_0, \ J_\pm] = \pm J_\pm$ and $[J_+, J_-] = 0$.

The dual action with respect to the rotational Killing vector field J_0 is

$$\tilde{S} = \int \partial\rho\bar{\partial}\rho + \frac{1}{\rho^2} \partial\varphi\bar{\partial}\varphi \ , \tag{A.4}$$

while φ itself transforms non–locally under T–duality,

$$\varphi \rightarrow \int \frac{1}{\rho^2} (dz\partial\varphi - d\bar{z}\bar{\partial}\varphi) \ . \tag{A.5}$$

The conformal invariance of the model determines the corresponding dilaton field, which is irrelevant for the present purposes. According to this, $J_\pm$ become non–locally realized in the dual formulation of our toy model. Only J_0 remains local in this case, generating with $J_\pm$ the same symmetry algebra as before.

The 2–dim theory (A.1) describes two free bosons and so it possesses a chiral $U(1) \times U(1)$ world–sheet current algebra. The two chiral currents in question are simply $\partial(\rho e^{\pm i\varphi})$ in polar coordinates. In the dual formulation they become non–locally realized and assume the form

$$\Psi_\pm = \partial \left(\rho e^{\pm i \int \frac{1}{\rho^2} (dz\partial\varphi - d\bar{z}\bar{\partial}\varphi)} \right) \ , \tag{A.6}$$

using eq. (A.5). Since they are chirally conserved, $\bar{\partial}\Psi_\pm = 0$, they are the parafermions of the dual model substituting the original $U(1) \times U(1)$ local currents. The parafermions of opposite chirality may be introduced in a similar way. The vector fields $J_\pm$ are also non–local in the dual formulation, but they do not deserve the name parafermions, because they are not chirally conserved.

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